Learning Enhancement Team

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STUDENT SUPPORT SERVICE

Steps into Calculus

Integrating $y = ax^n$

This guide describes how to integrate functions of the form $y = ax^n$. It introduces the power rule of integration and gives a method for checking your integration by differentiating back. This guide also discusses the special case of $n = -1$.

Introduction

Many problems in calculus involve functions of the form $y = ax^n$ and this general function covers **polynomials**, **reciprocal functions** and **roots**. The study guide: *Differentiating Using the Power Rule* introduced a method to find the derivative of these functions called **the power rule for differentiation**. This study guide is about integrating functions of the form $y = ax^n$ and takes a similar approach by introducing **the power rule for integration**. However there is a slight difference between the two approaches which you should be aware of, importantly the power rule for integration **does not work when** $n = -1$. The reason for this and its solution is discussed in this guide.

This guide only has examples of **indefinite integrals** (integrals without limits on the integral sign but with a "$+c$" at the end of the answers) but the same techniques apply to **definite integrals** (see study guide: *Definite Integrals*).

To help you get the most out of this guide a thorough understanding of the laws of indices and the rules of arithmetic involving fractions is essential – see study guides: *Laws of Indices*, *Adding and Subtracting Fractions* and *Multiplying and Dividing Fractions*. Also, as integration and differentiation are closely linked, you should also read the study guides: *Differentiating Using the Power Rule*, *Differentiating Basic Functions* and *What is Integration?*.

The power rule for integration

The power rule for the integration of a function of the form $y = ax^n$ is:

If $\quad y = ax^n \qquad (n \neq -1)$

then $\quad \int ax^n dx = a\dfrac{x^{n+1}}{n+1} + c$

The power rule works when *a* and *n* are any number (positive, negative, fraction or decimal) apart from when $n = -1$. If $n = -1$, you would be dividing by zero in the answer. Dividing by zero is not allowed in mathematics. In fact, $n = -1$ is a special case which requires its own rule and is discussed towards the end of this guide.

Indefinite integration can be thought of as the inverse operation to differentiation (see the study guide: *What is Integration?*). If you integrate a function and then differentiate it you return to the original function. Therefore, the power law for integration is the inverse of the power rule for differentiation which says:

If $\quad y = ax^n$

then $\quad \frac{dy}{dx} = anx^{n-1}$

and can be summarised as you subtract one from the power and multiply by the old power *n*. So, to integrate $y = ax^n$, you have to do the reverse: **add one to the power *n* and divide by the new power**.

It is common for the process of integration and differentiation to become confused i.e. you differentiate when you should integrate and vice-versa. However paying careful attention to the questions that are asked should help you to choose the correct process. The following might help you remember which one to do.

Differentiation	**D**ecreases the power	$\frac{dy}{dx} = anx^{n-1}$
Integration	**I**ncreases the power	$\int ax^n dx = a\frac{x^{n+1}}{n+1} + c$

Using the power rule for integration

As with the power rule for differentiation, to use the power rule for integration successfully you need to become comfortable with how the two parts of the power rule interact. To begin with, you must be able to identify those functions which can be (and just as importantly those which cannot be) integrated using the power rule. It may be necessary to manipulate a function in some way to see that it fits the pattern $y = ax^n$. When this is done, you need to be able to identify the values of *a* and *n*, substitute them into the second part of the rule and then simplify if necessary to give the answer.

In the following examples you will see a variety of functions which can be integrated using the power rule. The examples will cover a variety of different cases for *a* and *n*. All the examples will be indefinite integrals so you will see a constant *c* added to the answer.

Example: Integrate $y = 4x^2$ with respect to *x*.

When using the power rule for integration you must first ensure that the function you are integrating fits the pattern of the rule; in other words, can you correctly identify *a* and *n*? It may help to write the first part of the power rule for integration underneath the function you are integrating to help you see the pattern. Here:

$$y = 4x^2$$
$$y = ax^n$$

You can see that this fits the power rule for integration with $a = 4$ and $n = 2$, so inserting the values of *a* and *n* into the power rule gives:

$$\int ax^n dx = a\frac{x^{n+1}}{n+1} + c$$
$$\int 4x^2 dx = 4\frac{x^{2+1}}{2+1} + c = \frac{4x^3}{3} + c$$

Notice that the question "Integrate $y = 4x^2$ with respect to *x*" is written symbolically as:

$$\int 4x^2 dx$$

and the answer does not contain either an integral sign or *dx*. This is because the integral has been performed to give a function and they are no longer necessary. It is a common mistake for symbols to be either missing or added unnecessarily to questions involving integration so remember **integrals themselves have integration signs and a *dx* at the end but the results of integration (the answers) do not**.

Example (*with hidden coefficient*): Integrate $y = x^4$ with respect to *x*.

By writing the first part of the rule underneath the question can you see that $n = 4$?

$$y = \quad x^4$$
$$y = ax^n$$

However, the value of *a* is not obvious as there seems to be no number in front of the *x*. It is a common mistake to assume that $a = 0$ as it is absent, but this is *not* the case. If *a* was zero then the whole function would be zero (as anything multiplied by zero is zero). In fact the function can be rewritten as $y = 1 \cdot x^4$, and so $a = 1$. Using these values for *a* and *n* in the power rule for integration gives:

$$\int ax^n dx \quad = a\frac{x^{n+1}}{n+1} + c$$

$$\int 1 \cdot x^4 dx = 1 \cdot \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$$

These next two examples highlight the common functions $y = ax$ and $y = a$ both of which illustrate cases of $y = ax^n$ with *n* hidden.

Example: Integrate $y = 5x$ with respect to *x*.

Using the law of indices $x = x^1$, the function $y = 5x$ can be re-written as $y = 5x^1$. This fits the power rule with $a = 5$ and $n = 1$. Using these values for *a* and *n* in the power rule gives:

$$\int 5x dx = \int 5x^1 dx = 5 \cdot \frac{x^{1+1}}{1+1} + c = \frac{5x^2}{2} + c$$

Example: Integrate the constant $y = 3$ with respect to *x*.

Using the law of indices $x^0 = 1$, you can re-write the function $y = 3$ as $y = 3x^0$. So here you use $a = 3$ and $n = 0$ in the power rule for integration:

$$\int 3dx = \int 3x^0 dx = 3 \cdot \frac{x^{0+1}}{0+1} + c = \frac{3x^1}{1} + c = 3x + c$$

In all of the examples above, identifying *a* and *n*, and hence using the power rule to integrate the function, is relatively straightforward. The following example is a little more difficult and relies on both a good knowledge of the laws of indices and fractions to identify *a* and *n*.

Example: Integrate $y = \frac{\sqrt{x}}{7}$ with respect to *x*.

Upon first inspection this function does not look like the pattern $y = ax^n$. However when you interpret the “divide by 7” as a “multiply by $\frac{1}{7}$” and use the laws of indices to see that $\sqrt{x} = \sqrt[2]{x} = x^{1/2}$, the function becomes $y = \frac{1}{7}x^{1/2}$ which can be integrated using the power rule for integration with $a = \frac{1}{7}$ and $n = \frac{1}{2}$. So:

$$\int \frac{\sqrt{x}}{7} dx = \int \frac{1}{7} x^{\frac{1}{2}} dx = \frac{1}{7} \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{7} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2x^{\frac{3}{2}}}{21} + c$$

Integrating negative powers of *x* (other than $n = -1$)

As mentioned in the introduction, the power rule for integration works for all negative powers except $n = -1$. For this section it is useful to recall the law of indices $x^{-n} = 1/x^n$ which allows you to replace positive powers of x below a dividing line with a negative power of x.

Example: Integrate the function $y = \frac{6}{x^3}$ with respect to x.

Using the law of indices above you can re-write the function in question as $y = 6x^{-3}$ and so you can use the power law for integration with $a = 6$ and $n = -3$ to integrate the function:

$$\int \frac{6}{x^3}dx = \int 6x^{-3}dx = 6 \cdot \frac{x^{-3+1}}{-3+1} + c = 6 \cdot \frac{x^{-2}}{-2} + c = -3x^{-2} + c$$

Integrating the special case of $n = -1$

When $n = -1$, the power law for integration would mean adding 1 to -1 to give 0 and then dividing the answer by it. But in mathematics dividing by zero is an undefined operation and always means something is wrong. The solution to this problem lies in differentiation, remember that in the study guide: *Differentiating Basic Functions* the rule that:

If $y = a\ln x$

then $\frac{dy}{dx} = ax^{-1} = \frac{a}{x}$

So, because integration is the reverse of differentiation you can see that:

If $y = ax^{-1} = \frac{a}{x}$

then $\int \frac{a}{x}dx = \int ax^{-1}dx = a\ln x + c$

which is the rule to integrate functions when $n = -1$.

Example: Integrate the function $y = \frac{6}{x}$ with respect to x.

As the function can be written as $y = 6x^{-1}$ you can see that here $a = 6$ and $n = -1$ and you cannot use the power rule for integration. Instead use the above rule for when $n = -1$ to give:

$$\int \frac{6}{x}dx = \int 6x^{-1}dx = 6\ln x + c$$

Checking your integration

In general, integrating is harder than differentiating but, if you have a good understanding of differentiation, you can check to see if you have integrated correctly by **differentiating back**.

Remember, differentiating is the inverse of integrating and so, when you differentiate your answer, you should get the original function that you integrated.

Example: Confirm that $\int \frac{3}{4x^2}\,dx = -\frac{3}{4x} + c$.

You can confirm that the integral is correct by using the power rule of differentiation to see if the derivative of $-\frac{3}{4x} + c$ with respect to x is indeed $\frac{3}{4x^2}$. Differentiating:

$$\text{when} \quad y = -\frac{3}{4x} + c = -\frac{3}{4}x^{-1} + c$$

$$\text{then} \quad \frac{dy}{dx} = -\frac{3}{4}\cdot(-1)x^{-1-1} + 0 = \frac{3}{4}x^{-2} = \frac{3}{4x^2}$$

which is the function you integrated to begin with and so the statement is correct.

You can look back at the examples in this sheet and check them by differentiating back. You should differentiate back each time you perform an integral until you are confident with the technique.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in **Student Services**, as well as speaking to your lecturer or adviser.

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